14. Two parabolic arches are being built. The first arch can be modelled by the function $y = -x^2 + 9$, with a range of $0 \le y \le 9$. The second arch must span twice the distance and be translated 6 units to the left and 3 units down.

a) Sketch the graph of both arches.

b) Determine the equation of the second arch.

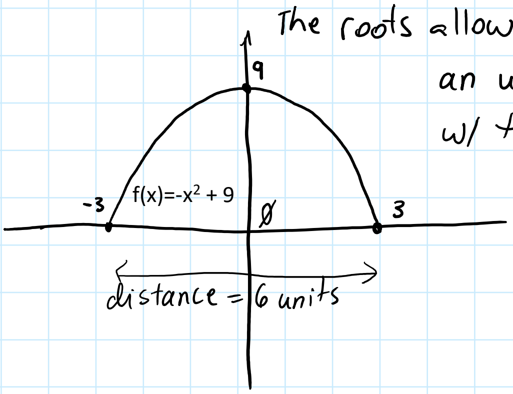
→ when told distance, we differentiate btwn distance and height, therefore this must be a horizontal issue.

(a) Sketching

diff. of squares

$f(x) = -x^2 + 9 = -(x^2 - 9) = -(x^2 - 3^2) = -(x-3)(x+3)$

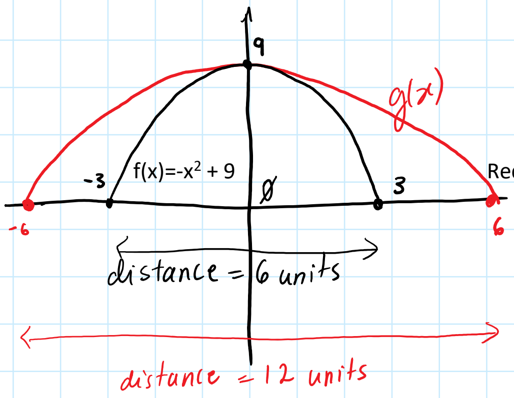
⇒ the 2 roots of $f(x)$ are: -3 & $+3$



The roots allow us to plot $f(x)$; it's an upside down parabola w/ these roots.

We need a second arch, first we only focus on spanning twice the distance.

↳ horizontal: otherwise it would be "twice the height" but that's not what we are told.



Red = twice the distance $= 2 \times 6 = 12 \to g(x)$

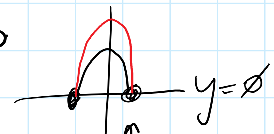
→ This is a stretch. There are two possibilities:

1) vertical stretch $g(x) = af(x)$

2) horizontal stretch $g(x) = f(bx)$

why can it not be a vertical stretch?

A vertical stretch changes the y coordinates and preserves the roots b/c $a \times \emptyset = \emptyset$



but our $g(x)$ graph doesn't looks like that and the red $g(x)$ above does not preserve the roots b/c the distance is known to double, and thus new roots formed at -6 and $+6$.

see example in Desmos:

https://www.desmos.com/calculator/2eiisl1pvg

$\Downarrow$

This cannot be a vertical stretch. It must be horizontal.

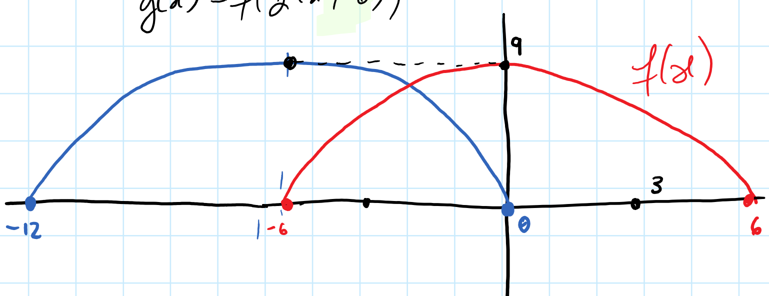
$\Rightarrow$ $g(x) = f(bx)$, & since we know it has stretched by a factor of 2, we conclude b is the inverse $b = \frac{1}{2}$

$\therefore$ $g(x) = f(\frac{1}{2}x) = -(\frac{1}{2}x)^2 + 9$

Let's continue w/the remaining transformations:

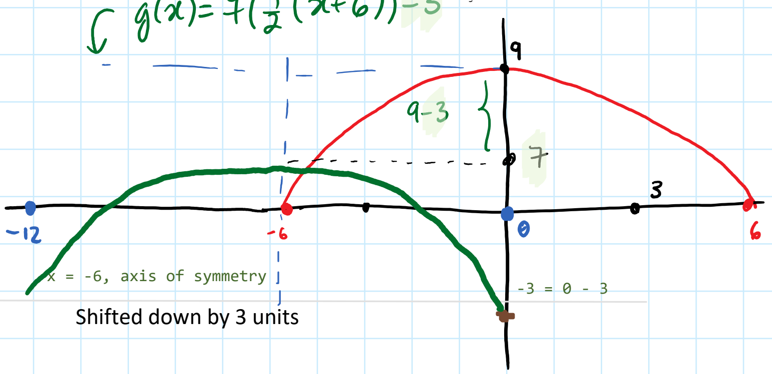
The red parabola after translating 6 units to the left, becomes $g(x)$, the blue parabola.

$g(x) = f(\frac{1}{2}(x+6))$



Blue = Red shifted to the left by 6

$g(x) = f(\frac{1}{2}(x+6)) - 3$



Shifted down by 3 units

This concludes the sketching of $g(x)$ & $f(x)$.

(b)

Based on the previous section:

$f(x) = -x^2 + 9$

$g(x) = f\left(\frac{1}{2}(x+6)\right) - 3 = -\left(\frac{1}{2}(x+6)\right)^2 + 9 - 3$

$= -\left(\frac{1}{2}(x+6)\right)^2 + 6$ ← textbook favours this form

open brackets

$= -\left(\frac{1}{2} \cdot x + \frac{1}{\cancel{2}_{1}} \cdot \cancel{6}^{3}\right)^2 + 6 =$

$= -\left(\frac{x}{2} + 3\right)^2 + 6 =$

$= -\left(\frac{x^2}{2^2} + \cancel{2} \cdot \frac{x}{\cancel{2}} \cdot 3 + 3^2\right) + 6 =$

$= -\left(\frac{x^2}{4} + 3x + 9\right) + 6 =$

$= -\frac{x^2}{4} - 3x - 9 + 6 =$

$= -\frac{1}{4}x^2 - 3x - 3$